Quantum Knizhnik–Zamolodchikov Equation: Reflecting boundary conditions and Combinatorics

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We consider the level 1 solution of quantum Knizhnik–Zamolodchikov equation with reflecting boundary conditions which is relevant to the Temperley–Lieb model of loops on a strip. By use of integral formulae we prove conjectures relating it to the weighted enumeration of Cyclically Symmetric Transpose Complement Plane Partitions and related combinatorial objects.

1. Introduction

Since the papers [1,2], there has been a great deal of work on the combinatorial interpretation of quantum integrable models at special points of their parameter space. The original observation is that the numbers of Alternating Sign Matrices (ASM) and Plane Partitions (PP) in various symmetry classes appear naturally in the ground state entries of the Temperley–Lieb $O(\tau = 1)$ model of (non-crossings) loops with various boundary conditions (and related models). The appearance of ASM numbers was developed further and to some extent explained by the Razumov–Stroganov conjecture [3] and variants [4,5] interpreting each ground state entry as a number of certain subsets of ASM. The role of plane partitions remained more obscure until the recent work [6,7] which showed that the enumeration of symmetry classes of PP also occurs naturally on condition that one consider a slightly more general problem, namely the quantum Knizhnik–Zamolodchikov equation (qKZ), first introduced in this context in [8], and in which the parameter τ is now free. This provided a (conjectural) bridge between enumerations of symmetry classes of ASM and PP, which is a fascinating topic of enumerative combinatorics in itself.

The present work is concerned more specifically with the case of the Temperley–Lieb loop model (and its qKZ generalization) defined on a strip with reflecting boundary conditions (the case of periodic boundary conditions was treated similarly in [9]). The corresponding ASM were discovered in [4,10]: they are Vertically Symmetric Alternating Sign Matrices (VSASM) of size $(2n+1) \times (2n+1)$ in even strip size $N = 2n$, and modified VSASM of size $(2n+1) \times (2n+3)$ in odd strip size $N = 2n+1$. As to the PP, they were discussed in [7]: they are Cyclically Symmetric Transpose Complement Plane Partitions (CSTCPP) [11] in odd strip size, and certain modified CSTCPP (referred to as CSTCPP$^\triangle$ in the following) in even strip size. The conjectures of [7] concerning the τ-enumeration of these plane partitions are the main subject of this work. In Sect. 2 we shall review the basics of integrable loop models based on the Temperley–Lieb algebra; in Sect. 3 we shall discuss the related qKZ equation, and review the conjectures of [7]; in Sect. 4 we introduce the main technical tool, that is certain explicit integrals solving qKZ; and finally in Sect. 5 and 6 we prove the conjectures of [7], considering separately even and odd cases.

2. Loop model with reflecting boundary conditions and link patterns

2.1. Dense Loop model on a strip

We consider the version with reflecting boundaries[1] of the inhomogeneous $O(1)$ non-crossing loop model [12]. The model is defined on a semi-infinite strip of width N (even or odd) of square lattice, with centers of the lower edges labelled $1, 2, ..., N$. On each face of this domain of the square lattice, we draw at random, say with respective probabilities $1 - t_i, t_i$ in the i-th column (at the vertical of the point labelled i) one of the two following configurations

$$
\hspace{3cm} \tag{2.1}
$$
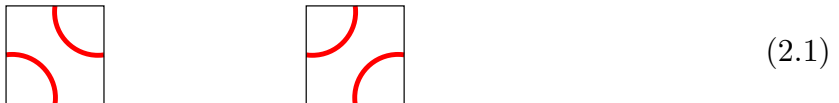
The strip is moreover supplemented with the following pattern of fixed configurations of loops on the (left and right) boundaries:

$$
\hspace{3cm} \tag{2.2}
$$
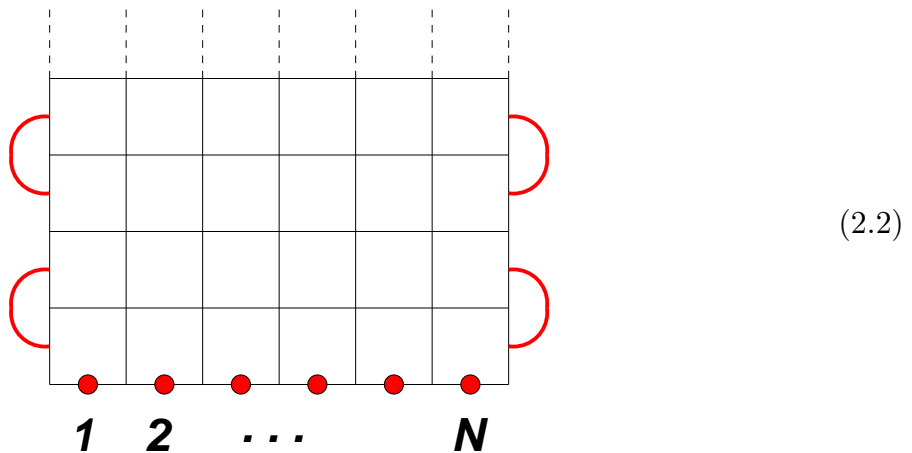
With probability 1, a configuration will lead to a pairing of the points $1, 2, ..., N$ according to their connection via the paths (except for one point if N is odd which is connected to the infinity along the strip). Such a pattern of connections is called a *link pattern*, and an individual pairing is called an arch. The set of link patterns on N points is denoted by LP_N, and has cardinality $(2n)!/(n!(n+1)!)$ for $N = 2n$ or $N = 2n - 1$. Each link pattern of odd size $N = 2n - 1$ may indeed be viewed as a link pattern of size $2n$ but with the point $2n$ sent to infinity on the strip: this provides a natural bijection between LP_{2n-1} and LP_{2n}. A link pattern $\pi \in LP_N$ may also be viewed as a permutation

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[1] These boundary conditions are sometimes called "open", in reference to the equivalent open XXZ spin chain, or "closed", due to the way the loops close at the boundaries of the strip.
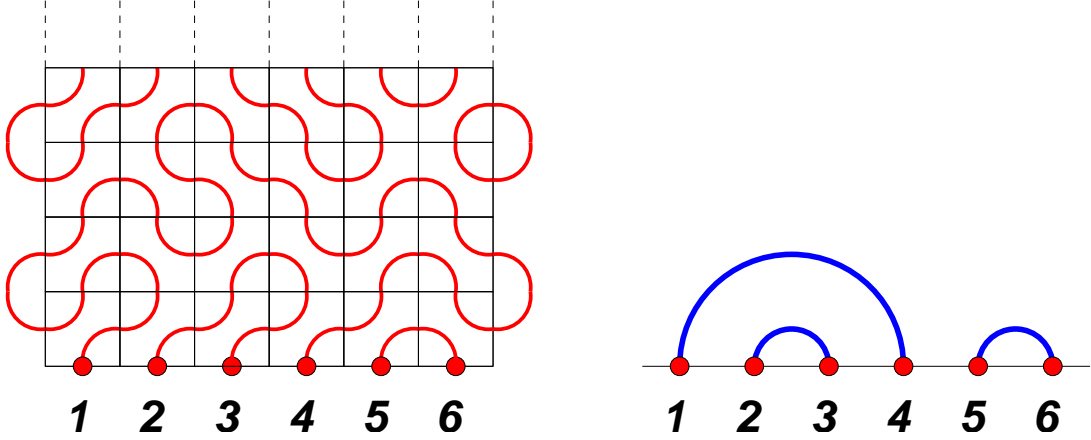
Fig. 1: A sample configuration of the Dense Loop model on a strip of width $N = 6$ (left). We have indicated the corresponding open non-crossing link pattern of connection of the points $1, 2, 3, 4, 5, 6$ (right).

$\pi \in S_N$ with only cycles of length 2 (except one cycle of length 1 for N odd), and we shall use the notation $\pi(i) = j$ to express that points i and j are connected by an arch. For a pair (i, j) such that $j = \pi(i)$ and $i < j$, we will call i the opening and j the closing of the arch connecting i and j. An example of loop configuration together with its link pattern are depicted in Fig. 1. We use a standard pictorial representation for link patterns in the form of configurations of non-intersecting arches connecting regularly spaced points on a line, within the upper-half plane it defines. For odd N, the unmatched point may be represented as connected to infinity in the upper-half plane via a vertical half-line. We moreover attach a weight $\tau = -(q + q^{-1}) = 1$ to each loop (hence the denomination $O(\tau = 1)$ model, $q = -e^{i\pi/3}$). We may then compute the probability $\text{Prob}(\pi)$ for a given randomly generated configuration of the loop model on the strip to be connecting the boundary points according to a given link pattern π.

In Ref. [12], the model was solved by means of a transfer matrix technique, using solutions of the boundary Yang–Baxter equation [13] [14] that parametrize the inhomogeneous probabilities t_i via integrable Boltzmann weights, coded by a standard trigonometric R-matrix. Using the integrability of the system, and following the philosophy of [15], the suitably renormalized vector of probabilities $\Psi \equiv \{\Psi_\pi\}_{\pi \in LP_N}$ was shown to satisfy the quantum Knizhnik–Zamolodchikov equation with reflecting boundaries for $q = -e^{i\pi/3}$, in the link pattern basis.

In the following, we will consider the more general case of generic q, τ, which does not have stricto sensu an interpretation in terms of lattice loop model [16].

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2.2. R matrix

The R-matrix of the model is an operator acting on link patterns of LP_N:

$$\check{R}_{i,i+1}(z,w) = \frac{qz - q^{-1}w}{qw - q^{-1}z} \bigg\langle\bigg|\bigg| \bigg\rangle + \frac{z-w}{qw - q^{-1}z} \bigg\langle\equiv\bigg\rangle$$

$$= \frac{qz - q^{-1}w}{qw - q^{-1}z} I + \frac{z-w}{qw - q^{-1}z} e_i \tag{2.3}$$

where z and w are complex numbers attached to the points i and $i+1$ and where e_i, $i = 1, ..., N-1$ are the generators of the Temperley-Lieb algebra $TL_N(\tau)$, subject to the relations

$$e_i^2 = \tau e_i, \qquad [e_i, e_j] = 0 \text{ if } |i-j| > 1, \qquad e_i e_{i\pm 1} e_i = e_i \tag{2.4}$$

with the parametrization

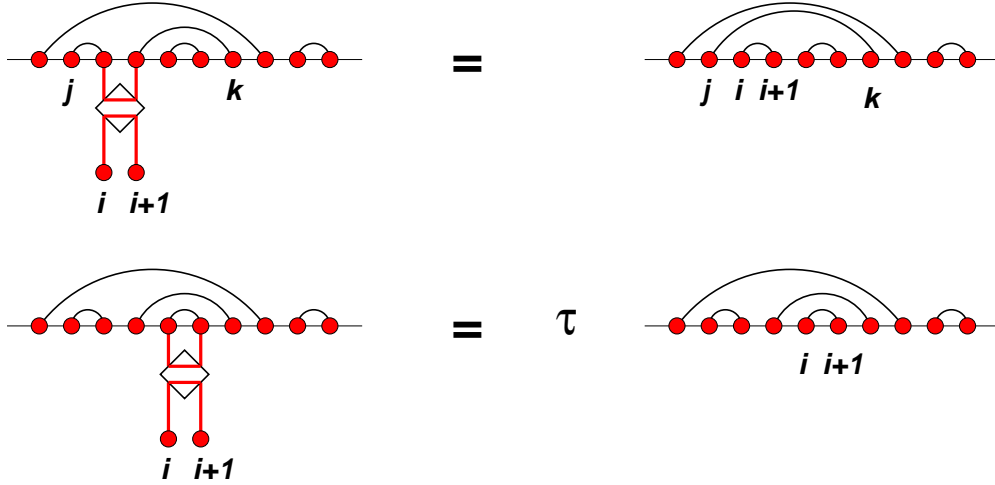$$\tau = -q - q^{-1} \tag{2.5}$$

Fig. 2: Action of the Temperley-Lieb generators e_i on link patterns.

In (2.3), we have depicted the Temperley-Lieb generators as tilted squares with edge centers connected by pairs. The corresponding action on link patterns should be understood as follows (see Fig. 2): assume the points $i, i+1$ are connected to say the points j, k in a link pattern π. Then, unless $j = i+1$ and $k = i$, the link pattern $\pi' = e_i\pi$ is identical to π except that i is now connected to $i+1$, and j to k. If $j = i+1$ and $k = i$, the points $i, i+1$ are connected to each other in π, and $e_i\pi = \tau\pi$. The latter is a direct consequence

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of the projector condition $e_i^2 = \tau e_i$, as any link pattern with i connected to $i+1$ lies in the image of e_i.

The above R-matrix satisfies the Yang–Baxter equation and the unitarity relation

$$\check{R}_{i,i+1}(z,w)\check{R}_{i+1,i+2}(z,x)\check{R}_{i,i+1}(w,x) = \check{R}_{i+1,i+2}(w,x)\check{R}_{i,i+1}(z,x)\check{R}_{i+1,i+2}(z,w)$$
$$\check{R}_{i,i+1}(z,w)\check{R}_{i,i+1}(w,z) = I \tag{2.6}$$

as consequencs of the Temperley-Lieb algebra relations (2.4).

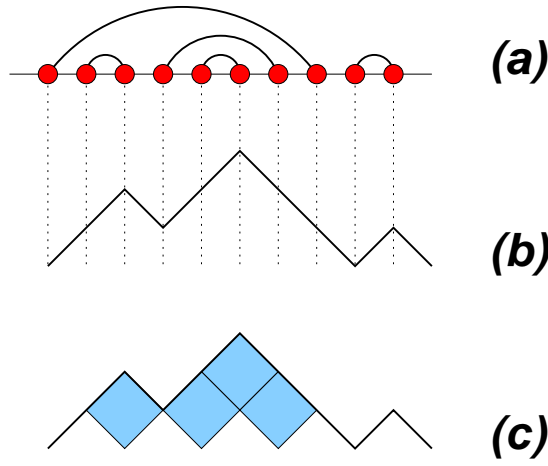*2.3. Link patterns, dyck paths, and containment order*

Fig. 3: Dyck path (b) associated to a link pattern (a). The former is obtained as the discrete path on the non-negative integer line: $0, 1, 2, 1, 2, 3, 2, 1, 0, 1, 0$. We have represented in (c) the box decomposition of the Dyck path.

Before turning to the qKZ equation, we wish to emphasize a number of useful properties satisfied by the link patterns, and the action of the Temperley-Lieb generators. An alternative pictorial representation of link patterns is via Dyck paths, namely paths from and to the origin on the non-negative integer line, with steps of ± 1 only. The bijection between link patterns and Dyck paths is illustrated on Fig. 3 for $N = 10$. For even $N = 2n$, we construct the Dyck path by visiting the points of the link pattern from 1 to N, starting from the origin of the non-negative integer half-line, and with the following rule: if an arch opens (resp. closes) at i, then the path goes up (resp. down) one step between time $i-1$ and i. The path is guaranteed to come back to the origin at time $2n$ as there are as many openings as closings of arches, and moreover stays in the non-negative half-line as all arches must first open before closing. In the case of odd $N = 2n - 1$, one arch exactly

has an opening and no closing point (it is connected to infinity), hence the path ends up at the point 1 on the integer half-line. It can be completed into a path of length $2n$ by a final step to the origin, thus expressing on Dyck paths the abovementioned bijection between LP_{2n-1} and LP_{2n}. The Dyck path is represented in the plane as the (broken-line) graph of the function $(t, h(t))$, $t = 0, 1, ..., N$.

Dyck paths allow to endow the set of link patterns with a natural "containment" order, namely $\pi < \pi'$ iff the Dyck path of π contains strictly that of π'. This notion is made explicit by introducing the "box decomposition" of any given Dyck path (see Fig. 3 (c)), namely the decomposition of the region of the plane delimited by the path and a broken line $(0, 0) \to (1, 1) \to (2, 0) \to \cdots \to (N-1, 1) \to (N, 0)$ if $N = 2n$, without the last step for $N = 2n - 1$, by use of squares of edge $\sqrt{2}$ tilted by $45°$. Then a Dyck path δ contains strictly another δ' iff δ is obtained from δ' by addition of at least one box. A box addition at position m consists simply in replacing a portion of path $(m-1, h) \to (m, h-1) \to (m+1, h)$ that visits successively the points $h, h-1, h$ of the integer half-line at times $m-1, m, m+1$, by the portion $(m-1, h) \to (m, h+1) \to (m+1, h)$, thus adding a box with center at coordinates (m, h). This may also be described as transforming a local minimum into a local maximum at position m on the path. The "smallest" link pattern (whose Dyck path contains all others) is the pattern π_0 with links $\pi_0(i) = 2n+1-i$, $i = 1, 2, ..., n$ for $N = 2n$, and $i = 2, ..., n$, for $N = 2n - 1$, while $\pi_0(1) = 1$. It corresponds to the farthest excursion, reaching point n on the integer half-line. The "largest" link pattern (whose Dyck path is contained in all others) $\pi_{\max}$ has $\pi_{\max}(i) = i + 1$ for $i = 1, 3, .., 2n - 1$ when $N = 2n$ and $i = 1, 3, ..., 2n - 3$ when $N = 2n - 1$, while $\pi_{\max}(2n - 1) = 2n - 1$. It corresponds to the shortest range excursion, alternating betwen the origin and point 1 on the integer half-line. So we have $\pi_0 < \pi < \pi_{\max}$ for all $\pi \in LP_N$ such that $\pi \neq \pi_0, \pi_{\max}$. Finally, we shall denote by $\beta(\pi)$ the total number of boxes in the box decomposition of the Dyck path associated to π. We have for instance $\beta(\pi_0) = n(n-1)/2$ and $\beta(\pi_{\max}) = 0$.

The action of e_i on link patterns may be easily translated into the language of boxes on Dyck paths. The action of e_i may indeed be viewed as a box addition at position i on the corresponding Dyck paths. Then 3 situations may occur (Fig. 4):

(i) The path has a minimum at point i: the box addition transforms it into a maximum.

(ii) The path has a maximum at point i: the box-added path is unchanged, but picks up a factor of τ.

(iii) The path has a slope at i, namely a succession of two up or two down steps: the box addition actually destroys the two rows of boxes at its height and immediately below
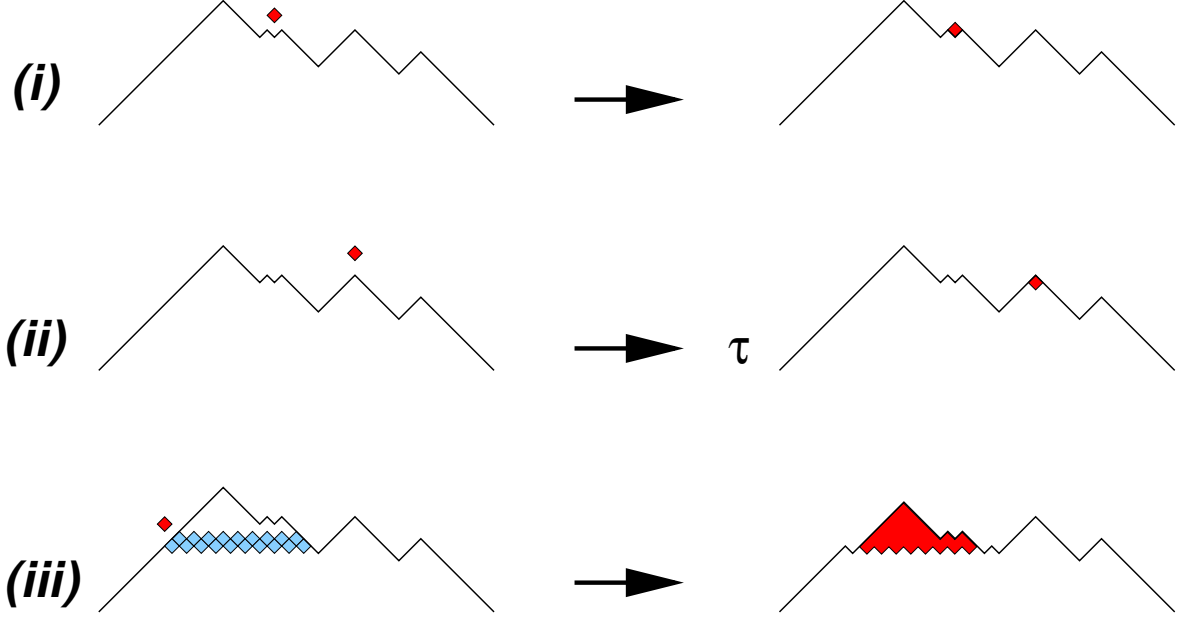
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Fig. 4: Box addition at position i on Dyck paths corresponding to the action of e_i. We have depicted three generic situations for the addition: (i) at a local minimum (ii) at a local maximum (iii) at a slope. Both (ii) and (iii) lead to a Dyck path contained by the original one, while (i) produces a Dyck path containing it, with exactly one additional box.

until the other side of the path is reached. The net result may be interpreted as an avalanche, in which the mountain shape between the point of impact and the other side falls down by two units.

This allows to see that among all possible actions of e_i on a link pattern π, only one leads to a "larger" Dyck path (containing π): $e_i\pi = \pi' < \pi$, namely in the situation (i), while all other situations lead to $\pi < \pi' = e_i\pi$. This observation will be used below.

The interpretation of the action of e_i on Dyck paths was used in [5] to rephrase the homogeneous loop model as the stochastic model of a growing interface.

3. The qKZ equation for reflecting boundary condition

3.1. The equation

The reflecting boundary qKZ equation consists of the following system of equations for a vector Ψ which depends polynomially on the variables $z_1, \ldots, z_N$ (and q, q^{-1}):

$$\check{R}_i(z_{i+1}, z_i)\Psi_N(z_1, \ldots, z_i, z_{i+1}, \ldots, z_N) = \Psi_N(z_1, \ldots, z_{i+1}, z_i, \ldots, z_N) \qquad (3.1a)$$

$$c_N(z_N)\Psi_N(z_1, \ldots, z_N) = \Psi_N(z_1, \ldots, z_{L-1}, s/z_N) \qquad (3.1b)$$

$$c_1(z_1)\Psi_N(z_1, \ldots, z_N) = \Psi_N(1/z_1, z_2, \ldots, z_N) \qquad (3.1c)$$

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Here c_1 and c_N are scalar functions to be specified later, and $s = q^{2(k+2)}$ is a parameter which determines the "level" k of the equation: here we consider the so-called level 1 case, namely with $s = q^6$.

One can think of Eqs. (3.1) as an analogue of the quantum Knizhnik–Zamolodchikov equation (qKZ) in the form introduced by Smirnov [17] (see also [18]), in which one replaces the periodic boundary conditions, implicit in the usual qKZ, with reflecting boundaries [14]. More precisely, Eq. (3.1a) is the exchange relation corresponding to the bulk, independent of boundary conditions, whereas Eqs. (3.1b,c) implement the reflections at the two boundaries.

In [16], it was remarked that solving these equations for even size $N = 2n$ automatically provides a solution for odd size $N - 1$ by taking the last parameter z_N to zero (or equivalently to infinity). We therefore discuss in detail the case of even size now, postponing to Sect. 6 the discussion of the odd case.

3.2. Minimal polynomial solution

In [12], it was claimed that the system of equations (3.1) possesses a polynomial solution of minimal degree $3n(n-1)$ which is unique up to multiplication by a scalar. To actually solve the equations (3.1), one first remarks that the $N - 1$ equations (3.1a) from a triangular system with respect to the containment order of Dyck paths introduced in Sect. 2.3. Indeed, when written in components, this equation reads:

$$\frac{q^{-1}z_{i+1} - qz_i}{z_{i+1} - z_i}(\tau_i - 1)\Psi_\pi(z_1, \ldots, z_{2n}) = \sum_{\substack{\pi':\pi'\neq\pi \\ e_i\pi'=\pi}} \Psi_{\pi'}(z_1, \ldots, z_{2n}) \tag{3.2}$$

where τ_i acts on functions of the z's by interchanging z_i and z_{i+1}. Now consider the sum on the r.h.s.: it extends over the proper inverse images of π under e_i. Picking π in the image of e_i (i.e. with an arch connecting points i and $i+1$, as explained above), its inverse images π' under e_i all have dyck paths containing that of π (i.e. $\pi' < \pi$) except one, say π^*, corresponding to the Dyck path of π with the box at position i removed, hence with $\pi < \pi^*$. Hence Eq. (3.2) allows to express Ψ_{π^*} in terms of only Ψ_α, with $\alpha < \pi^*$. The solution is therefore uniquely fixed by specifying the component corresponding to the smallest link pattern π_0 defined above, whose Dyck path that contains all others. The

latter is entirely fixed by the degree condition and factorization properties deduced from the qKZ system; the result is:

$$\Psi_{\pi_0} = \prod_{1 \leq i < j \leq n} (q\, z_i - q^{-1} z_j)(q^2 - z_i z_j) \prod_{n+1 \leq i < j \leq 2n} (q\, z_i - q^{-1} z_j)(q^4 - z_i z_j) \qquad (3.3)$$

This fixes the functions c_1 and c_N in Eqs. (3.1b, c) to be $c_1(x) = 1/x^{2n-2}$ and $c_N(x) = (q^3/x)^{2n-2}$.

It is also a simple exercise to prove that the solution enjoys a reflection invariance property, inherited from the inversion (unitarity) relation satisfied by the R-matrix (2.6), and clearly satisfied by the fundamental component Ψ_{π_0}:

$$\Psi_\pi(z_1, \ldots, z_{2n}) = \prod_{i=1}^{2n} \left(\frac{z_i}{q^3}\right)^{n-1} \Psi_{\rho(\pi)}\left(\frac{q^3}{z_{2n}}, \ldots, \frac{q^3}{z_1}\right) \qquad (3.4)$$

where $\rho(\pi)$ is the mirror image of π w.r.t. a vertical axis.

At the special value $q = -\exp(i\pi/3)$, which is a "degenerate" case of the qKZ system where the shift s becomes unity, the vector Ψ can be interpreted as the ground state eigenvector of the loop model described in Sect. 2. For generic q, no such direct physical interpretation of Ψ exists; however, it still retains remarkable combinatorial properties that we shall describe now.

3.3. Conjectures

In Ref. [7], a number of conjectures were made on the homogeneous, generic q limit of the components of Ψ. The dependence on q was always expressed through the quantity $\tau = -q - q^{-1}$, and the entries of Ψ were observed to be polynomials of τ with non-negative integer coefficients. The main conjectures concerned an identification of the sum of components of Ψ with generating polynomials for weighted rhombus tilings of finite domains of the triangular lattice, involving the catalytic variable τ. These were interpreted in odd and even sizes as τ-enumerations of respectively Cyclically Symmetric Transpose Complementary Plane Partitions (CSTCPP), or equivalently of rhombus tilings of a hexagon with cyclic and transpose-complementary symmetries (odd size), and in the even case as modified CSTCPP (CSTCPP$^\triangle$) corresponding to rhombus tilings of a hexagon with a central triangular hole and with cyclic and transpose-complementary symmetries. These conjectures were extensions of earlier conjectures concerning only the particular value $\tau = 1$ (q cubic root of unity), due to Razumov and Stroganov [4], and proved in [12], involving

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respectively the total number of CSTCPP in odd size, and that of Vertically Symmetric Alternating Sign Matrices (VSASM) in even size.

Apart from the main sum rule conjectures, a number of other conjectures were made in [7] on the entries of Ψ as polynomials of τ, concerning degree and valuation, and also conjecturally relating the small τ behavior to the numbers of Totally Symmetric Self-Complementary Plane Partitions (TSSCPP) with fixed features, namely, once expressed as Non-Intersecting Lattice Paths (NILP), with fixed termination points of the paths.

The purpose of the present paper is to prove these various conjectures, by means of multiple integral expressions for the homogeneous solution to the qKZ equation $\Psi(\tau)$.

4. Integral expressions for solutions of level 1 qKZ

4.1. The intermediate basis

The method introduced in [9] in order to obtain integral representations of Ψ was to exhibit a different basis than that of link patterns in which the integral expressions for the components are relatively simple. We shall use the same basis here. Note that this section is "boundary conditions-independent" and its results are equally valid for say periodic boundary conditions.

The elements of the basis we consider are indexed by strictly increasing integer sequences of the form $\mathbf{a} \equiv \{a_1, a_2, \ldots, a_n\}$, where $1 \leq a_i \leq 2i - 1$. We denote by O_n the set of such sequences. The sequences in O_n are in one-to-one correspondence with the link patterns in LP_{2n} in two different, inequivalent ways. One may indeed associate to each $\pi \in LP_{2n}$ the sequence $a_i(\pi)$, $i = 1, 2, \ldots, n$, of the positions (counted from left to right, and taking values in $\{1, 2, \ldots, 2n - 1\}$) of the openings of arches in π. Similarly, we may associate to π the sequence $b_i(\pi) = a_i(\rho(\pi))$, $i = 1, 2, \ldots, n$ recording the closing positions of the arches of π, counted from right to left, or equivalently the opening positions of the arches in the reflection $\rho(\pi)$.

In [9], we have constructed the change of basis from the link pattern basis to the "arch opening" basis, namely expressed the solutions Ψ_π, $\pi \in LP_{2n}$ in terms of multiple residue integrals $\Psi_{a_1, \ldots, a_n}$, with $\{a_1, a_2, \ldots, a_n\} \in O_n$. More precisely, Ref. [9] expresses any integral $\Psi_{\mathbf{a}} \equiv \Psi_{a_1, \ldots, a_n}$ for weakly increasing sequences of a's as well, in terms of the solution Ψ_π in the link pattern basis, via the linear transformation:

$$\Psi_{\mathbf{a}}(z_1, \ldots, z_{2n}) = \sum_{\pi \in LP_{2n}} C_{\mathbf{a}, \pi}(\tau) \Psi_\pi(z_1, \ldots, z_{2n}) \tag{4.1}$$

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with polynomial coefficients $C_{\mathbf{a},\pi}(\tau)$ expressed as follows. Let $U_k \equiv (q^{k+1} - q^{-k-1})/(q - q^{-1})$. For $k \geq 0$, U_k is the Chebyshev polynomial of the second kind associated to $-\tau$, defined recursively by $U_{k+1} = -\tau U_k - U_{k-1}$ with $U_0 = 1$ and $U_{-1} = 0$. A given link pattern $\pi \in LP_{2n}$ may alternatively be thought of as a permutation of $\{1, 2, \ldots, 2n\}$ with only cycles of length 2. The arches forming π may therefore be described by the pairs $(i, \pi(i))$, $1 \leq i \leq 2n$, such that $i < \pi(i)$, in which case i are the positions of the openings and $\pi(i)$ of the closings of the arches in π. Then we have the following formula:

$$C_{\mathbf{a},\pi}(\tau) = \prod_{\substack{i=1 \\ i<\pi(i)}}^{2n-1} U_{\mu(\mathbf{a},i)} \tag{4.2}$$

where

$$\mu(\mathbf{a}, i) = \mathrm{card}\,\{j | i \leq a_j < \pi(i)\} + \frac{i - \pi(i) - 1}{2} \tag{4.3}$$

In [9], it was mentioned that the entries (4.2) may easily be computed by iteratively removing the "little arches" of π (with $\pi(i) = i + 1$), and replacing them with a factor U_{m-1}, where m is the total number of a's lying under that arch (namely such that $a_j = i$). The new link pattern thus obtained has one less arch, and its a's are relabeled accordingly, while $m - 1$ extra a's are placed in position $i - 1$. The algorithm is iterated until the link pattern becomes empty.

If we moreover restrict the set of a's to O_n, we get a true change of basis from link patterns to arch openings, in which $C(\tau)$ is a square invertible $c_n \times c_n$ matrix with polynomial coefficients. The matrix $C(\tau)$ indeed enjoys the following properties. Let us first use the bijection between LP_{2n} and O_n to write $C_{\mathbf{a},\pi}(\tau) \equiv C_{\alpha,\pi}(\tau)$, where $\alpha \in LP_{2n}$ is uniquely determined by its arch opening positions $a_1, a_2, \ldots a_n$, counted from left to right. Let us moreover order the link patterns, say by increasing lexicographic order on the sets of positions of their arch openings. Then we have the property

(P) $C(\tau)$ is a lower triangular matrix, with entries 1 on the diagonal, and polynomials of τ with integer coefficients elsewhere, and the same holds for $C^{-1}(\tau)$.

Property (P) is easily derived as follows. First, it is clear that the diagonal terms $C_{\mathbf{a}(\pi),\pi} = 1$, as $\mu(\mathbf{a}(\pi), i) = 0$ for all arch openings i of π: indeed, the a'a being the arch openings of π, the set $\{a_j | i \leq a_j < \pi(i)\}$ has one a per arch enclosed by the arch $i \to \pi(i)$, henceforth a total of $(\pi(i) - i + 1)/2$. Consequently, all the indices of the Chebyshev polynomials contributing to (4.2) vanish, and as $U_0 = 1$, the result follows. The triangularity is best understood by following the abovementioned algorithm for constructing C. Indeed, at any

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step in the algorithm, the only cause for the matrix element to vanish is that one has no a's under the little arch considered, as in this case one would get a factor $U_{-1} = 0$. The matrix element $C_{\alpha,\pi}$ can only be non-zero if the arch openings of α occupy positions lexicographically larger that those of π. Indeed, by contradiction, assume the structure of arch openings in α and π are the same up to say a position i where an arch opens in α while an arch closes in π. This means that the total number of arch openings for positions $j > i$ is strictly larger in π than in α. Consequently, applying the above algorithm to the arches of π opening at positions $> i$, we see that at least one little arch in the process will have no arch opening of α below it, thus receiving a weight $U_{-1} = 0$, and therefore the corresponding matrix element $C_{\alpha,\pi}$ vanishes.

Finally, one can rewrite the qKZ equation itself using the linear combinations defined by Eq. (4.1). Note here that we are forced to use not only the components corresponding to our basis O_N of increasing sequences, but also those corresponding to any non-decreasing sequence. In principle all of them can be reexpressed as linear combinations of increasing sequences only, but it is preferrable to avoid having to write these linear dependence relations explicitly.

All that is needed is the action of the e_i on the $\Psi_{\mathbf{a}}$. We have the following

Theorem 1: *For any non-decreasing sequence $a_1, \ldots, a_n$ such that the number i occurs exactly k times, $k \geq 0$, we have the formula:*

$$(e_i\Psi)_{a_1,\ldots,\underbrace{i,\ldots,i}_{k},\ldots,a_n} = U_{k-1}U_{k-4}\Psi_{a_1,\ldots,a_n} - U_{k-1}U_{k-3}\big(\Psi_{a_1,\ldots,i-1,\underbrace{i,\ldots,i}_{k-1},\ldots,a_n}$$

$$+ \Psi_{a_1,\ldots,\underbrace{i,\ldots,i}_{k-1},i+1,\ldots,a_n}\big) + U_{k-1}U_{k-2}\Psi_{a_1,\ldots,i-1,\underbrace{i,\ldots,i}_{k-2},i+1,\ldots,a_n} \quad (4.4)$$

(where for $k = 0$ the r.h.s. is zero).

Proof: expand the l.h.s. in the basis of link patterns by using Eq. (4.1). We find:

$$(e_i\Psi)_{\mathbf{a}} = \sum_{\pi:\pi(i)\neq i+1} C_{\mathbf{a},\mathbf{e_i}(\pi)}\Psi_\pi + \tau \sum_{\pi:\pi(i)=i+1} C_{\mathbf{a},\pi}\Psi_\pi \quad (4.5)$$

where we have distinguished among the entries of e_i its diagonal entries, equal to τ, and its non-diagonal entries, equal to 1.

The Ψ_π must be regarded here as independent objects, so that we must now check Eq. (4.4) for each link pattern π. This will be performed by a case by case analysis of the situation around the sites $(i, i+1)$. Each time only the coefficients involving the arches

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starting or ending at $i, i+1$ differ from term to term in the equation, so that we can ignore the remaining factors. The proof will be explained pictorially using the same conventions as in appendix 1 of [9], that is by drawing the coefficient $C_{\mathbf{a},\pi}$ as the usual (local) depiction of the link pattern π decorated by placing between sites i and $i+1$ (inside a circle) the total number k of a's such that $a_j = i$.

There are 4 cases:

(i) If i is an opening and $i+1$ a closing of π, then π has a little arch $(i, i+1)$: $\pi(i) = i+1$. In this case the equality reduces pictorially to

or explicitly

$$\tau U_{k-1} = U_{k-1}U_{k-4} \times U_{k-1} - U_{k-1}U_{k-3} \times 2U_{k-2} + U_{k-1}U_{k-2} \times U_{k-3} \qquad (4.6)$$

which is easily checked by noting that $U_{k-1}U_{k-4} - \tau = U_{k-2}U_{k-3}$.

In all other cases there is no little arch $(i, i+1)$.

(ii) If both i and $i+1$ are openings, call p the total "weight" under the arch leaving $i+1$, that is $p = \text{card}\{\ell | i+1 \leq a_\ell < \pi(i+1)\} - \frac{i+1-\pi(i+1)-1}{2}$, and q the remaining weight under the bigger arch starting from i, excluding what is under the smaller arch and the weight k under the segment $[i, i+1)$, in order to make the pictorial description simpler: $q = \text{card}\{\ell | \pi(i+1) \leq a_\ell < \pi(i)\} - \frac{\pi(i) - \pi(i+1) - 1}{2}$. Then the identity to prove is:

$$U_{k-1}U_{q-1} = U_{k-1}U_{k-4} \times U_{p-1}U_{k+p+q-2} - U_{k-1}U_{k-3} \times U_{p-1}U_{k+p+q-3}$$
$$- U_{k-1}U_{k-3} \times U_pU_{k+p+q-2} + U_{k-1}U_{k-2} \times U_pU_{k+p+q-3} \qquad (4.7)$$

which is again a routine check.

(ii') The case where i and $i+1$ are both closings is treated analogously.

(iii) Finally, if i is a closing and $i+1$ an opening, call p the weight under the arch $(\pi(i), i)$ defined as before, and q the weight under the arch $(i+1, \pi(i+1))$. Similary the proof of the identity

$$U_{k-1}U_{k+p+q-2} = U_{k-1}U_{k-4} \times U_{p-1}U_{q-1} - U_{k-1}U_{k-3} \times U_pU_{q-1}$$
$$- U_{k-1}U_{k-3} \times U_{p-1}U_q + U_{k-1}U_{k-2} \times U_pU_q \qquad (4.8)$$

is left to the reader. This completes the proof of the theorem 1.

4.2. Integral solution of qKZ equation: general principle

The idea to use integral representations for solutions of the qKZ equation is not new and there is a vast literature on the subject (cf the references in Sect. 11.2 of [19]). We consider here a very specific type of level 1 solutions, for which one expects a much simpler formula than generically. In the present context, this idea was used in [9] in the case of the qKZ equation with the usual periodic boundary conditions. We now describe the procedure in a slightly more general (boundary conditions-independent) setting.

The idea is to define for any non-decreasing sequence $(a_1, \ldots, a_n)$ the following quantity:

$$\Psi_{a_1, \ldots, a_n}(z_1, \ldots, z_N) = \prod_{1 \le i < j \le N} (qz_i - q^{-1}z_j) \oint \cdots \oint \prod_{m=1}^{n} \frac{dw_\ell}{2\pi i}$$
$$F(z_1, \ldots, z_N; w_1, \ldots, w_n) \frac{\prod_{1 \le \ell < m \le n}(w_m - w_\ell)(qw_\ell - q^{-1}w_m)}{\prod_{\ell=1}^{n} \prod_{i=1}^{a_\ell}(w_\ell - z_i) \prod_{i=a_\ell+1}^{N}(qw_\ell - q^{-1}z_i)} \qquad (4.9)$$

where F is any rational function that is symmetric in all z_i and symmetric in all w_ℓ. The contours of integration encircle the z_i but not $q^{-2}z_i$, nor any poles of F.

We wish to prove that (4.9) solves the exchange relation (3.2) of the qKZ equation, in the operatorial form $t_i\Psi = (e_i - \tau)\Psi$ where the action of e_i is given by Eq. (4.4), and we have introduced he divided difference operator $t_i = (qz_i - q^{-1}z_{i+1})\partial_i$. As before, we have to

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consider the cases where exactly k a's take the value i, say $a_m = a_{m+1} = \cdots = a_{m+k-1} = i$, while $a_{m-1} < i$ or $m = 1$ and $a_{m+k} > i$ or $m + k = N$. Starting from the expression (4.4), and subtracting $\tau\Psi$ on both sides, we wish to express $(e_i - \tau)\Psi$ as the action on Ψ of the divided difference operator t_i. We are left with proving the following

Theorem 2: *The function (4.9) solves the exchange relation of the qKZ equation, namely it satisfies:*

$$
\begin{aligned}
(t_i\Psi)_{a_1,\ldots,a_{m-1},\underbrace{i,\ldots,i}_{k},a_{m+k},\ldots,a_n} &= U_{k-2}U_{k-3}\Psi_{a_1,\ldots,a_{m-1},\underbrace{i,\ldots,i}_{k},a_{m+k},\ldots,a_n} \\
&- U_{k-1}U_{k-3}\Big(\Psi_{a_1,\ldots,a_{m-1},i-1,\underbrace{i,\ldots,i}_{k-1},a_{m+k},\ldots,a_n} + \Psi_{a_1,\ldots,a_{m-1},\underbrace{i,\ldots,i}_{k-1},i+1,a_{m+k},\ldots,a_n}\Big) \\
&+ U_{k-1}U_{k-2}\Psi_{a_1,\ldots,a_{m-1},i-1,\underbrace{i,\ldots,i}_{k-2},i+1,a_{m+k},\ldots,a_n}
\end{aligned}
\tag{4.10}
$$

Proof: Two important remarks are in order. Firstly, the operator t_i acts only on the pieces of Ψ that are non-symmetric in (z_i, z_{i+1}). When acting with t_i on (4.9), we may restrict our attention to the non-symmetric part of the integrand. Secondly, we note that for any function $S(u_1, \ldots, u_k)$ satisfying the following vanishing antisymmetrizer property that

$$
\mathcal{A}(S) \equiv \sum_{\sigma \in S_k} (-1)^\sigma S(u_{\sigma(1)}, \ldots, u_{\sigma(k)}) = 0
\tag{4.11}
$$

then the multiple integral

$$
I_k \equiv \oint du_1 \cdots du_k S(u_1, \ldots, u_k) \prod_{1 \leq \ell < m \leq k} (u_m - u_\ell) = 0
\tag{4.12}
$$

Indeed, for any given permutation $\sigma \in S_k$, we may perform the change of variables $u_m = v_{\sigma(m)}$ for $m = 1, 2, \ldots, k$, resulting in $I_k = \oint d\mathbf{v} S(\mathbf{v}^\sigma)\Delta(\mathbf{v}^\sigma) = \oint d\mathbf{v}(-1)^\sigma S(\mathbf{v}^\sigma)\Delta(\mathbf{v})$, where we have used the antisymmetry of the Vandermonde determinant, and finally summing over all permutations yields (4.12), thanks to (4.11). We also note that if S satisfies (4.11), any symmetric function of the u's multiplied by S will also satisfy it.

As a consequence of the two above remarks, to prove the identity (4.10), it is sufficient to prove a weaker statement on the part P of the integrand of Ψ that is non-symmetric in (z_i, z_{i+1}) and also non-symmetric in $(w_m, w_{m+1}, \ldots, w_{m+k-1})$, after factoring out a

Vandermonde determinant of the w's. Rewriting $u_i = w_{m+i-1}$ for $i = 1, 2, \ldots, k$, P reads simply

$$P_k \equiv (qz_i - q^{-1}z_{i+1}) \frac{\prod_{1 \leq \ell < m \leq k}(qu_\ell - q^{-1}u_m)}{\prod_{\ell=1}^{k}(u_\ell - z_i)(qu_\ell - q^{-1}z_{i+1})} \tag{4.13}$$

Now introducing the quantity

$$S(u_1, \ldots, u_k) \equiv t_i P_k - \left(U_{k-2}U_{k-3} + U_{k-1}U_{k-3}(f_1 + g_k) - U_{k-1}U_{k-2}f_1 g_k \right) P_k \tag{4.14}$$

where we use the notations

$$f_\ell = \frac{u_\ell - z_i}{qu_\ell - q^{-1}z_i} \qquad g_\ell = \frac{qu_\ell - q^{-1}z_{i+1}}{u_\ell - z_{i+1}} \tag{4.15}$$

for $\ell = 1, 2, \ldots, k$, we are simply left with the task of proving that (4.11) is indeed satisfied, and (4.10) will follow from the above considerations.

For simplicity, we will work with the quantity $T = S \times \frac{z_{i+1}-z_i}{q\,z_i - q^{-1}z_{i+1}} \prod_{\ell=1}^{k}(u_\ell - z_i)(q\,u_\ell - q^{-1}z_{i+1})$, proportional to S of (4.14) by a factor symmetric in the u's, henceforth proving $\mathcal{A}(S) = 0$ amounts to proving $\mathcal{A}(T) = 0$. Explicitly, we have

$$\mathcal{A}\Big(T(u_1, \ldots, u_k)\Big) = \mathcal{A}\Big(\Delta_q(\mathbf{u})\Big) \left((qz_{i+1} - q^{-1}z_i) \prod_{\ell=1}^{k} f_\ell g_\ell - (qz_i - q^{-1}z_{i+1}) \right)$$
$$- (z_{i+1} - z_i)\mathcal{A}\Big(\Delta_q(\mathbf{u})(U_{k-2}U_{k-3} - U_{k-1}U_{k-3}(f_1 + g_k) + U_{k-1}U_{k-2}f_1 g_k)\Big) \tag{4.16}$$

where we have written for short the q-Vandermonde as $\Delta_q(\mathbf{u}) \equiv \prod_{1 \leq \ell < m \leq k}(qu_\ell - q^{-1}u_m)$ and noticed that only this piece of P is non-symmetric in the u's.

We need the following lemma, expressing the antisymmetrization of the q-Vandermonde:

$$\mathcal{A}\Big(\Delta_q(\mathbf{u})\Big) = (-1)^{k(k-1)/2} U_1 U_2 \cdots U_{k-1} \, \Delta(\mathbf{u}) \tag{4.17}$$

Proof. We proceed by induction. For $k = 2$, an explicit computation leads to $\mathcal{A}(qu_1 - q^{-1}u_2) = (q + q^{-1})(u_1 - u_2) = -U_1\Delta(\mathbf{u})$. Assume (4.17) holds for $k \to k-1$. We decompose any permutation $\sigma \in S_k$ according to the image of 1, say $\sigma(1) = m$. Upon relabelling indices, the corresponding permutation σ' is in S_{k-1}, and we may apply the recursion hypothesis to the summation over such σ'. We have

$$\mathcal{A}\Big(\Delta_q(\mathbf{u})\Big) = \sum_{m=1}^{k} \sum_{\substack{\sigma \in S_k \\ \sigma(1)=m}} (-1)^\sigma \Delta_q(\mathbf{u}^\sigma)$$
$$= (-1)^{k(k-1)/2} \sum_{m=1}^{k} \prod_{\substack{\ell=1 \\ \ell \neq m}}^{k} \frac{qu_m - q^{-1}u_\ell}{u_m - u_\ell} U_1 U_2 \cdots U_{k-2}\Delta(\mathbf{u}) \tag{4.18}$$

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where we have applied the recursion hypothesis to $\Delta_q(u_1, \ldots, u_{m-1}, u_{m+1}, \ldots, u_k)$ and reabsorbed the sign change by $(-1)^{k-1}$ into the ratio $\Delta(\mathbf{u})/\prod_{\ell \neq m}(u_m - u_\ell)$. To conclude, we still have to prove the following sublemma:

$$\varphi_k(u_1, \ldots, u_k) \equiv \sum_{m=1}^{k} \prod_{\substack{\ell=1 \\ \ell \neq m}}^{k} \frac{qu_m - q^{-1}u_\ell}{u_m - u_\ell} = U_{k-1} \tag{4.19}$$

for all distinct complex numbers $u_1, \ldots, u_k$. To prove the latter, let us first note that it is a rational fraction, symmetric in the u's. Viewed as a function of u_1, it has possible poles at $u_2, u_3, \ldots, u_k$ and is bounded at infinity. By symmetry, it is sufficient to compute the residue at $u_1 \to u_2$, for which only the two first terms in the summation contribute, leading to:

$$\mathrm{Res}_{u_1 \to u_2} \left(\frac{qu_1 - q^{-1}u_2}{u_1 - u_2} \prod_{\ell=3}^{k} \frac{qu_1 - q^{-1}u_\ell}{u_2 - u_\ell} - \frac{qu_2 - q^{-1}u_1}{u_1 - u_2} \prod_{\ell=3}^{k} \frac{qu_2 - q^{-1}u_\ell}{u_2 - u_\ell} \right) = 0 \tag{4.20}$$

Hence the function φ_k is bounded and has no pole in u_1, hence is independent of u_1, but as it is symmetric, it is a constant, say C_k. To compute it, we take the limit $u_1 \to \infty$, and find the recursion relation $\varphi_k(u_1, \ldots, u_k) = q^{k-1} + q^{-1}\varphi_{k-1}(u_2, \ldots, u_k)$, henceforth $C_k = q^{k-1} + q^{-1}C_{k-1}$. Moreover, by direct inspection, we find $C_1 = 0$, therefore the sequence C is entirely fixed, and coincides with that of the Chebyshev polynomials of the second kind, namely $C_k = U_{k-1} = (q^k - q^{-k})/(q - q^{-1})$. This completes the proof of (4.17).

Applying (4.17) to (4.16), and decomposing when necessary the permutation σ according to the images of 1 and/or k, say $\sigma(1) = j$ and $\sigma(k) = m$, we arrive at

$$\frac{\mathcal{A}\Big(T(u_1, \ldots, u_k)\Big)}{(-1)^{k(k-1)/2}U_1 \cdots U_{k-1}\Delta(\mathbf{u})} = \left((qz_{i+1} - q^{-1}z_i)\prod_{\ell=1}^{k} f_\ell g_\ell - (qz_i - q^{-1}z_{i+1}) \right)$$

$$- (z_{i+1} - z_i)\Bigg\{ U_{k-2}U_{k-3} - U_{k-3}\sum_{j=1}^{k} f_j \prod_{\substack{\ell=1 \\ \ell \neq j}}^{k} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} - U_{k-3}\sum_{m=1}^{k} g_m \prod_{\substack{\ell=1 \\ \ell \neq m}}^{k} \frac{qu_\ell - q^{-1}u_m}{u_\ell - u_m}$$

$$+ \sum_{1 \leq j \neq m \leq k} f_j g_m \frac{qu_j - q^{-1}u_m}{u_j - u_m} \prod_{\substack{\ell=1 \\ \ell \neq j,m}}^{k} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} \frac{qu_\ell - q^{-1}u_m}{u_\ell - u_m} \Bigg\}$$

$$\tag{4.21}$$

where we use the notations (4.15). Our last task is to prove that the r.h.s. of (4.21) vanishes identically (we denote it by B in the following). To this end, we view it as a

rational fraction of the variable z_{i+1}, with possible poles at $z_{i+1} \to u_s$, $\ell = 1, 2, \ldots, k$ and at infinity. We first compute the residue at $z_{i+1} \to u_s$. From the definition (4.15), we have $\mathrm{Res}_{z_{i+1} \to u_s}(g_s) = g'_s = -(q - q^{-1})u_s$, and all other terms have a finite limit, henceforth:

$$
\frac{1}{g'_s} \mathrm{Res}_{z_{i+1} \to u_s}(B) = (qu_s - q^{-1}z_i) \prod_{\ell=1}^{k} f_\ell \prod_{\substack{\ell=1 \\ \ell \neq s}}^{k} \frac{qu_\ell - q^{-1}u_s}{u_\ell - u_s}
$$

$$
- (u_s - z_i) \left\{ \sum_{\substack{j=1 \\ j \neq s}}^{k} f_j \prod_{\substack{\ell=1 \\ \ell \neq s,j}}^{k} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} - U_{k-3} \right\} \prod_{\substack{\ell=1 \\ \ell \neq s}}^{k} \frac{qu_\ell - q^{-1}u_s}{u_\ell - u_s} \tag{4.22}
$$

$$
= (u_s - z_i) \prod_{\substack{\ell=1 \\ \ell \neq s}}^{k} \frac{qu_\ell - q^{-1}u_s}{u_\ell - u_s} \left\{ \prod_{\substack{\ell=1 \\ \ell \neq s}}^{k} f_\ell - \sum_{\substack{j=1 \\ j \neq s}}^{k} f_j \prod_{\substack{\ell=1 \\ \ell \neq s,j}}^{k} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} + U_{k-3} \right\}
$$

Noting that the last bracket involves only functions of the $k-1$ variables $u_1, \ldots, u_{s-1}, u_{s+1}, \ldots, u_k$, its vanishing is actually the consequence of the following lemma, valid for all $p \geq 1$:

$$
\prod_{\ell=1}^{p} f_\ell - \sum_{j=1}^{p} f_j \prod_{\substack{\ell=1 \\ \ell \neq j}}^{p} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} + U_{p-2} = 0 \tag{4.23}
$$

Proof. Viewed as a function of z_i via the definition (4.15), the l.h.s. of (4.23) (which we denote by D) is a rational fraction, with possible poles at $z_i \to q^2 u_\ell$, $\ell = 1, 2, \ldots, p$ and at infinity. Let us first compute the residue at $z_i \to q^2 u_s$, for which the only contributions come from $\mathrm{Res}_{z_i \to q^2 u_s}(f_s) = (q - q^{-1})u_s = f'_s$:

$$
\frac{1}{f'_s} \mathrm{Res}_{z_i \to q^2 u_s}(D) = \prod_{\substack{\ell=1 \\ \ell \neq s}}^{p} \frac{u_\ell - q^2 u_s}{q(u_\ell - u_s)} - \prod_{\substack{\ell=1 \\ \ell \neq s}}^{p} \frac{qu_s - q^{-1}u_\ell}{u_s - u_\ell} = 0 \tag{4.24}
$$

So D has no finite pole, and it is moreover bounded at infinity, with limit

$$
\lim_{z_i \to \infty}(D) = q^p - q \sum_{j=1}^{p} \prod_{\substack{\ell=1 \\ \ell \neq p}}^{p} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} + U_{p-2} \tag{4.25}
$$

Applying the above sublemma (4.19), we may rewrite this into $q^p - qU_{p-1} + U_{p-2} = 0$. We conclude that $D = 0$, and henceforth B has no finite pole in z_{i+1}. We must now examine possible residues at $z_{i+1} \to \infty$. The leading behavior of B when $z_{i+1} \to \infty$ is polynomial

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of degree ≤ 1. Noting that all $\lim_{z_{i+1} \to \infty} g_\ell = q^{-1}$, the coefficient of z_{i+1} of B in this limit reads:

$$B|_{z_{i+1}} = q^{-1} + q^{1-k} \prod_{\ell=1}^{k} f_\ell - \left\{ \sum_{j=1}^{k} q^{-1} f_j \prod_{\substack{\ell=1 \\ \ell \neq j}}^{k} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} \sum_{\substack{m=1 \\ m \neq j}}^{k} \prod_{\substack{\ell=1 \\ \ell \neq j,m}}^{k} \frac{qu_\ell - q^{-1}u_m}{u_\ell - u_m} \right.$$

$$\left. -U_{k-3} \sum_{j=1}^{k} \left(f_j \prod_{\substack{\ell=1 \\ \ell \neq j}}^{k} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} + q^{-1} \prod_{\substack{\ell=1 \\ \ell \neq j}}^{k} \frac{qu_\ell - q^{-1}u_j}{u_\ell - u_j} \right) + U_{k-2}U_{k-3} \right\} \tag{4.26}$$

Applying the sublemma (4.19) with $k \to k - 1$ in the first line and k in the second, this simplifies into

$$B|_{z_{i+1}} = q^{-1} + q^{1-k} \prod_{\ell=1}^{k} f_\ell - (q^{-1}U_{k-2} - U_{k-3}) \sum_{j=1}^{k} f_j \prod_{\substack{\ell=1 \\ \ell \neq j}}^{k} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} - U_{k-3}(U_{k-2} - q^{-1}U_{k-1}) \tag{4.27}$$

After noting that $q^{-1}U_{k-2} - U_{k-3} = q^{1-k}$ and $U_{k-2} - q^{-1}U_{k-1} = -q^{-k}$, we finally apply the lemma (4.23) with $p = k$, with the result

$$B|_{z_{i+1}} = q^{-1} - q^{1-k}U_{k-2} + q^{-k}U_{k-3} = q^{-k}(q^{k-1} + U_{k-3} - qU_{k-2}) = 0 \tag{4.28}$$

Hence we conclude that B has no finite pole in z_{i+1} and is bounded at $z_{i+1} \to \infty$, it is therefore independent of z_{i+1}. We now evaluate B at $z_{i+1} = 0$, in which case all $g_\ell = q$, and

$$B|_{z_{i+1}=0} = -z_i \left(q + q^{k-1} \prod_{\ell=1}^{k} f_\ell - (qU_{k-2} - U_{k-3}) \sum_{j=1}^{k} f_j \prod_{\substack{\ell=1 \\ \ell \neq j}}^{k} \frac{qu_j - q^{-1}u_\ell}{u_j - u_\ell} - U_{k-3}(U_{k-2} - qU_{k-1}) \right) \tag{4.29}$$

where we have used again the sublemma (4.19). Again, we note that $qU_{k-2} - U_{k-3} = q^{k-1}$ and $U_{k-2} - qU_{k-1} = -q^k$, and we apply the lemma (4.23) with $p = k$ to get

$$B|_{z_{i+1}=0} = -z_i(q - q^{k-1}U_{k-2} + q^k U_{k-3}) = -z_i q^k (q^{1-k} + U_{k-3} - q^{-1}U_{k-2}) = 0 \tag{4.30}$$

We may therefore conclude that $B = 0$ identically, which implies that $\mathcal{A}(T) = 0 = \mathcal{A}(S)$, which in turn implies (4.10), as explained above. This completes the proof of Theorem 2.

4.3. Case of reflecting boundary conditions

All that has been described above can for example apply to the case of periodic boundary conditions treated in [9], avoiding the lengthy discussion found in this paper; in this case the function F is just 1. In the present case of reflecting boundaries, the solution to the qKZ equation must incorporate the new boundary conditions, cf Eqs. $(3.1b, c)$, which correspond to a non-trivial choice of F, namely:

$$\Psi_{a_1,\ldots,a_n}(z_1,\ldots,z_N) = \prod_{1\le i<j\le N}(qz_i - q^{-1}z_j)(q^4 - z_iz_j) \oint \cdots \oint \prod_{m=1}^{n} \frac{dw_\ell}{2\pi i}$$

$$\frac{\prod_{1\le \ell<m\le n}(w_m - w_\ell)(qw_\ell - q^{-1}w_m)(q^2 - w_\ell w_m)\prod_{1\le \ell\le m\le n}(q^4 - w_\ell w_m)}{\prod_{\ell=1}^{n}\prod_{i=1}^{N}(q^4 - w_\ell z_i)\prod_{i=1}^{a_\ell}(w_\ell - z_i)\prod_{i=a_\ell+1}^{N}(qw_\ell - q^{-1}z_i)} \quad (4.31)$$

where we recall that the contours of integration encircle the z_i but not $q^{-2}z_i$, nor $q^4 z_i^{-1}$. By a computation of residues similar to what is performed in [15], it is easy to show that $\Psi_{a_1,\ldots,a_n}$ is in fact a polynomial in the variables $z_1,\ldots,z_N$ of degree $3n(n-1)$.

We now want to show that these $\Psi_{\mathbf{a}}$ identify with the components in the intermediate basis of the solution of the qKZ equation with reflecting boundary conditions discussed in Sect. 3.1. We note that both quantities satisfy the main equation (4.10) (exchange relation) of the qKZ system, as a consequence of Theorems 1 and 2 above. Furthermore, direct computation shows that when $a_i = i$ for $i = 1, 2, ..., n$, the only non-vanishing contribution to the integral (4.31) comes from the multiresidue at $w_i \to z_i$, for $i = 1, 2, ..., n$. Cancelling all denominators, we are left with the polynomial:

$$\Psi_{1,2,\ldots,n} = \prod_{1\le i<j\le n}(q\,z_i - q^{-1}z_j)(q^2 - z_iz_j) \prod_{n+1\le i<j\le 2n}(q\,z_i - q^{-1}z_j)(q^4 - z_iz_j) \quad (4.32)$$

which is nothing but Ψ_{π_0} (3.3), and indeed the change of basis implies that $\Psi_{1,2,\ldots,n} = \Psi_{\pi_0}$. Moreover, introducing the lexicographic order on non-decreasing sequences $(a_1,\ldots,a_n)$, we note that the sequence $(1, 2, ..., n)$ is the smallest yielding a non-zero result for the integral (4.31). Indeed, for any strictly smaller sequence, at least two residues will have to be taken at identical points say $w_\ell, w_m \to z_i$, causing the result to vanish, due to the Vandermonde determinant of the w's in the numerator.

The final step is to show that every component $\Psi_{\mathbf{a}}$ can be deduced from $\Psi_{1,2,\ldots,n}$ by use of Eq. (4.10). With respect to the lexicographic order on the non-decreasing sequences $(a_1,\ldots,a_n)$, Eq. (4.10) can be considered as a triangular linear system, allowing to compute $\Psi_{a_1,\ldots,a_{m-1},\underbrace{i,\ldots,i}_{k-1},i+1,a_{m+k},\ldots,a_n}$ in terms of other $\Psi_{\mathbf{a}}$ with smaller index. It is easy to conclude from this that the $\Psi_{\mathbf{a}}$ are entirely determined by the non-zero component with smallest index (4.32).

4.4. Homogeneous limit

Let us now evaluate the homogeneous limit of the components of Ψ, by setting $z_1 = z_2 = \cdots = z_{2n} = 1$, and by renormalizing it in such a way that $\Psi_{\pi_0} = 1$. This amounts to taking the integral formula (4.31), substituting $z_i = 1$ for all i, and dividing it out by $\Psi_{\pi_0}(1, 1, ..., 1) = q^{3n(n-1)/2}(q - q^{-1})^{2n(n-1)}(q + q^{-1})^{n(n-1)/2}$.

Performing then the change of variables $w_i = \frac{1-qu_i}{1-q^{-1}u_i}$ in the resulting expresion, we get:

$$J_{a_1,...,a_n}(\tau) = \tau^{n(2n-1)} \oint \cdots \oint \prod_{m=1}^{n} \frac{du_m}{2\pi i u_m^{a_m}} \left[\frac{\prod_{1\leq \ell \leq m \leq n}(\tau + (\tau^2 - 1)(u_\ell + u_m) + \tau(\tau^2 - 2)u_\ell u_m)}{\prod_{m=1}^{n}(\tau + (\tau^2 - 1)u_m)^{2n}} \right.$$

$$\left. \times \prod_{1\leq \ell < m \leq n} (u_m - u_\ell)(1 + \tau u_m + u_\ell u_m)(1 + \tau(u_\ell + u_m) + (\tau^2 - 1)u_\ell u_m) \right]$$

Note that the integration contours encircle 0 but not $U_2/U_1 = (1 - \tau^2)/\tau$. This integral formula is quite cumbersome and proves quite difficult to deal with. Instead, let us use the inversion property (3.4) to express the homogeneous components as the reflected components $\Psi_{\rho(\pi)}$ at $z_1 = z_2 = \cdots = z_{2n} = q^3$. This simplifies drastically the pole structure of (4.31), and finally, after division by $\Psi_{\pi_0}(1, 1, ..., 1)$, leads to the following integrals:

$$K_{b_1,...,b_n}(\tau) = \oint \cdots \oint \prod_{\ell=1}^{n} \frac{du_\ell}{2\pi i u_\ell^{b_\ell}} \left[\prod_{1\leq \ell \leq m \leq n} (1 - u_\ell u_m) \right.$$

$$\left. \times \prod_{1\leq \ell < m \leq n} (u_m - u_\ell)(1 + \tau u_m + u_\ell u_m)(\tau + u_\ell + u_m) \right] \qquad (4.33)$$

obtained by performing the change of variables $w_i = q\frac{1-qu_i}{1-q^{-1}u_i}$. We have the relation $J_{\mathbf{a}(\pi)} = K_{\mathbf{b}(\pi)}$ for all $\pi \in LP_{2n}$, hence we may call the K's the "arch closing" basis elements.

The expression in brackets in formula (4.33) can be be thought of as a (polynomial) "quasi-generating function" for the $K_{\mathbf{b}}$, that is the homogeneous limit of the $\Psi_{\mathbf{a}}/\Psi_{1,2...,n}$. An important consequence of this formula is that the $K_{\mathbf{b}}$ are polynomials with integer coefficients in τ. Since the change of basis from the intermediate basis to the basis of link patterns has entries that are also polynomials with integer coefficients in τ, we conclude that the homogeneous $\Psi_\pi(\tau)$, normalized by $\Psi_{\pi_0}(\tau) = 1$, possess the same property, as had been conjectured in [7]. In particular, at $\tau = 1$, where the $\Psi_\pi(1)$ are identified with

the ground state components of the loop model with reflecting boundaries normalized by $\Psi_{\pi_0}(1) = 1$, we conclude that all these components are integers, as had been conjectured earlier in [4].

Note that the same integrality argument applies equally well to the case of periodic boundary conditions treated in [9].

5.   Even case: proofs of various conjectures

5.1. Solution at $\tau \to 0$

When $\tau \to 0$, the integral (4.33) is easily evaluated upon changing to variables $v_m = u_m/\tau$ and explicitly retaining only the leading terms when $\tau \to 0$ in each factor. This results in

$$K_{b_1,\ldots,b_n}(\tau) \sim \tau^{n^2 - \sum b_i} \oint \cdots \oint \prod_{m=1}^{n} \frac{dv_m}{2\pi i v_m^{b_m}} \prod_{1 \leq \ell < m \leq n} (v_m - v_\ell)(1 + v_\ell + v_m) \qquad (5.1)$$

Upon identifying the product as the Vandermonde determinant $\Delta(v(1+v))$ (using the notation $\Delta(z) \equiv \prod_{1 \leq \ell < m \leq n}(z_m - z_\ell)$ for the Vandermonde determinant of $z \equiv \{z_1, \ldots, z_n\}$, also equal to $\det(z_\ell^{m-1})_{\ell, m=1,2,\ldots,n}$), we may recast (5.1) into a single determinant:

$$K_{b_1,\ldots,b_n}(\tau) \sim \tau^{n^2 - \sum b_i} \det_{1 \leq \ell, m \leq n} \left(\oint \frac{dv}{2\pi i v} v^{m-b_\ell}(1+v)^{m-1} \right)$$

$$\sim \tau^{n^2 - \sum b_i} \det_{1 \leq \ell, m \leq n} \binom{m-1}{b_\ell - m} \qquad (5.2)$$

The latter determinant is nothing but the number $N_{10}(b_1, b_2, \ldots, b_n)$ of NILP in bijection with TSSCPP with fixed endpoints $b_1, b_2, \ldots, b_n$ in their NILP formulation, while the power of τ reads $n(n-1) - \sum(b_i - 1) = \beta(\pi)$, which is also the number of boxes in the box decomposition of the Dyck path associated to π.

This result may now be immediately translated into an estimate for the small τ behavior of the qKZ solution in the link pattern basis. Indeed, the change of basis (4.1) allows to identify $\Psi_\pi(\tau) \sim K_{\mathbf{b}(\pi)}(\tau)$ when $\tau \to 0$. This is readily seen by writing

$$\Psi_\pi(\tau) = \sum_\alpha C^{-1}(\tau)_{\pi,\alpha} K_{\mathbf{b}(\alpha)}(\tau) \qquad (5.3)$$

and recalling that $C^{-1}(\tau)$ has all entries polynomial in τ, and that it is lower triangular with respect to containment order of the Dyck paths associated to the link patterns, we

deduce that any π such that $C^{-1}(\tau)_{\pi,\alpha}$ is non-zero must be contained in α, hence have a strictly smaller number of boxes if it is distinct from α. As $K_{\mathbf{b}(\alpha)}(\tau)$ behaves like $\tau^{n(n-1)-\beta(\alpha)}$, we deduce that any contribution to the sum (5.3) with $\alpha \neq \pi$ is subleading, as $\beta(\alpha) < \beta(\pi)$.

This completes the proof of the small τ conjecture of Ref. [7], in the case of even size, namely that

$$\Psi_\pi(\tau) \sim \tau^{\beta(\pi)} \, N_{10}\Big(b_1(\pi), \ldots, b_n(\pi)\Big) \tag{5.4}$$

where $b_i(\pi)$ denote the positions of the arch closures of π, counted from right to left.

5.2. Solution at large τ

For large τ, we obtain the leading contribution to $K_{b_1,\ldots,b_n}(\tau)$ by changing variables to $v_m = \tau u_m$ in the integral formula (4.33), and retaining only the leading order in τ within each factor in the integrand. This yields

$$K_{b_1,\ldots,b_n}(\tau) \sim \tau^{\sum(b_i-1)} \oint \cdots \oint \prod_{\ell=1}^n \frac{dv_\ell}{2\pi i v_\ell^{b_\ell}} \Delta(v) \prod_{\ell=1}^n (1+v_\ell)^{\ell-1} \tag{5.5}$$

By multilinearity of the Vandermonde determinant, this may be recast into

$$K_{b_1,\ldots,b_n}(\tau) \sim \tau^{\sum(b_i-1)} \det_{1\leq\ell,m\leq n} \left(\oint \frac{dv}{2\pi i v} v^{m-b_\ell}(1+v)^{\ell-1} \right)$$

$$\sim \tau^{\sum(b_i-1)} \det_{1\leq\ell,m\leq n} \binom{\ell-1}{b_\ell-m} \tag{5.6}$$

To translate this into a result for $\Psi_\pi(\tau)$, let us again consider the change of basis (5.3), and note that $\sum(b_i(\pi)-1) = n(n-1) - \beta(\pi)$. We wish to prove that $\Psi_\pi(\tau) \sim K_{\mathbf{b}(\pi)}(\tau)$ at large τ.
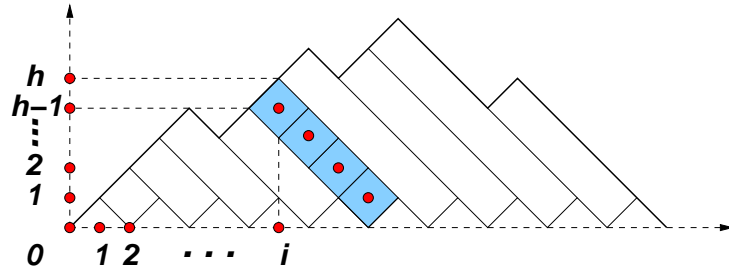

Fig. 5: The strip decomposition of a typical Dyck path. To each ascending step $(i-1, h-1) \to (i, h)$ we associate a diagonal row of $h-1$ boxes as indicated, thus forming a strip of boxes. Strips associated to ascending steps exhaust all the boxes in the decomposition of the Dyck path.

The degree of the matrix element $C_{\alpha,\pi}(\tau)$ in τ is given by the following quantity. Define first $h(\pi, \alpha)$ as the sum over the arch openings of α of the total number of arches of π sitting above their position (an arch $(i, \pi(i))$ is said to sit above position j iff $i \leq j < \pi(i)$). The quantity $h(\pi, \alpha)$ is also the sum over the heights $h_i(\pi)$ in the Dyck path of π (or equivalently the position occupied by the path on the integer half-line at time i), measured at the positions i of the points in the Dyck path of α reached by an ascending step (i.e. such that $h_i(\alpha) = h_{i-1}(\alpha) + 1$), namely:

$$h(\pi, \alpha) = \sum_{\substack{i=1 \\ h_i(\alpha)=h_{i-1}(\alpha)+1}}^{2n} h_i(\pi) \tag{5.7}$$

With this expression, it is easy to see that $h(\pi, \pi) = \beta(\pi) + n$ for any $\pi \in LP_{2n}$. Indeed, $h(\pi, \pi)$ is the sum of heights of ends of ascending steps in the Dyck path of π. As illustrated in Fig.5, we may associate to each such ascending step $(i - 1, h_{i-1}(\pi)) \rightarrow (i, h_i(\pi))$ with $h_i(\pi) = h_{i-1}(\pi) + 1$ the diagonal strip of $h_i(\pi) - 1$ boxes with centers at $(i + \ell - 1, h_i(\pi) - \ell)$, $l = 1, 2, ..., h_i(\pi) - 1$, and this exhausts all boxes of π. Such a "strip decomposition" was considered in Ref. [20]. We deduce that $\beta(\pi) = \sum_{i:h_i(\pi)=h_{i-1}(\pi)+1}(h_i(\pi) - 1) = h(\pi, \pi) - n$ as there are exactly n ascending and n descending steps in the Dyck path. Then, using the definition (4.2) and the fact that the Chebyshev polynomials U_m have degree m in τ, we have

$$d_{\alpha,\pi} \equiv \deg\left(C_{\alpha,\pi}(\tau)\right) = h(\pi, \alpha) - h(\pi, \pi) \tag{5.8}$$

Finally we need the following lemma:

The quantity $f_\pi(\alpha) = h(\pi, \alpha) + h(\alpha, \alpha)$, where α runs over the link patterns whose Dyck path is included in that of π, reaches its maximum at $\alpha = \pi$ only.

Proof: let us show that $f_\pi(\alpha)$ is a non-decreasing function with the size of α, namely the number of boxes in the decomposition of its Dyck path. Assume $\alpha' \in LP_{2n}$ differs from α by a single box say at positions $i-1, i, i+1$ in the Dyck path formulation, with identical heights $h_j(\alpha') = h_j(\alpha) + 2\delta_{j,i}$ except at the position i. We see easily that $h(\alpha', \alpha') = h(\alpha, \alpha) + 1$ as the ascending step $(i, i+1)$ in α is replaced by $(i - 1, i)$ in α', and $h_i(\alpha') = h_{i+1}(\alpha) + 1$. Moreover,

$$h(\pi, \alpha') = h(\pi, \alpha) + h_i(\pi) - h_{i+1}(\pi) \geq h(\pi, \alpha) - 1 \tag{5.9}$$

as $h_i(\pi) \geq h_{i+1}(\pi) - 1$. We deduce that $f_\pi(\alpha') \geq f_\pi(\alpha)$, hence that $f_\pi(\pi) \geq f_\pi(\alpha)$ for all α whose Dyck path is included in that of π. Finally, π is the unique maximum of $f_\pi(\alpha)$,

as is easily seen by removing a box from π say at position i and comparing heights in the Dyck paths. Indeed, we have say $(h_{i-1}(\pi), h_i(\pi), h_{i+1}(\pi)) = (m, m + 1, m)$, and in the box-removed π' we have $(h_{i-1}(\pi'), h_i(\pi'), h_{i+1}(\pi')) = (m, m - 1, m)$. Hence $h(\pi, \pi') = h(\pi, \pi) - (m + 1) + m$ and $h(\pi', \pi') = h(\pi, \pi) - (m + 1) + m$, so that $f_\pi(\pi') = f_\pi(\pi) - 2$.

We deduce from the above lemma an upper bound on the degree $d_{\alpha,\pi}$ (5.8): $d_{\alpha,\pi} = h(\pi, \alpha) - h(\pi, \pi) < h(\pi, \pi) - h(\alpha, \alpha) = \beta(\pi) - \beta(\alpha)$. We may now prove that the large τ contribution to $\Psi_\pi(\tau)$ is given by that to $K_{\mathbf{b}(\pi)}(\tau)$. This is done by induction on the (decreasing) number of boxes in α. Assume it is true for all π with a strictly larger number of boxes than α. Then for all these π, $\Psi_\pi(\tau)$ has degree equal to that of $K_{\mathbf{b}(\pi)}(\tau)$, namely $n(n - 1) - \beta(\pi)$. Then the expression $C_{\alpha,\pi}(\tau)\Psi_\pi(\tau)$, for $\alpha \subset \pi$ and $\pi \neq \alpha$ has degree $d_{\alpha,\pi} + n(n - 1) - \beta(\pi) < n(n - 1) - \beta(\alpha)$ by the above inequality. But $K_{\mathbf{b}(\alpha)}(\tau) = \sum_\pi C_{\alpha,\pi}(\tau)\Psi_\pi(\tau)$ has degree $n(n-1) - \beta(\alpha)$, hence it must be attained by the term $\pi = \alpha$ in the sum, and we deduce that $\Psi_\alpha(\tau) \sim K_{\mathbf{b}(\alpha)}(\tau)$ for large τ. As the result holds trivially for the largest link pattern π_0 (the only non-vanishing matrix element of C with this first entry is just $C_{\pi_0,\pi_0}(\tau) = 1$), this completes the desired proof.

5.3. Generalized sum rule

The fundamental remark of Ref. [9] for the even case was that summing $\Psi_{\mathbf{a}}$ over a specific subset of "opening arch" basis elements, namely the set of arch openings a's such that $a_i = 2i - 1 - \epsilon_i$, $\epsilon_i \in \{0, 1\}$, amounted to summing Ψ_π over the whole set LP_{2n}. This was readily seen as a property of the change of basis $C_{\alpha,\pi}(\tau)$ (4.2). Due to a reflection symmetry property, an analogous statement may be derived for the "arch closing" basis elements. As a result, the above sum rule for $\sum_{\pi \in LP_{2n}} \Psi_\pi$ is obtained by summing the integrals $K_{b_1, b_2, \ldots, b_n}(\tau)$ over the $b_i = 2i - 1 - \epsilon_i$, with $\epsilon_i \in \{0, 1\}$.

More precisely, let us consider

$$K(t|\tau) \equiv \sum_{\epsilon_i \in \{0,1\}} t^{\sum \epsilon_i} K_{1-\epsilon_1, 3-\epsilon_2, \ldots, 2n-1-\epsilon_n}(\tau) \tag{5.10}$$

This quantity has many interesting specializations: (i) $t = 0$ corresponding to the "maximum" component $K_{1,3,\ldots,2n-1}(\tau) = \Psi_{\pi_{max}}(\tau)$, where π_{max} connects the points $(2i - 1, 2i)$ via little arches only (ii) $t \to \infty$ corresponding to $K_{1,2,4,\ldots,2n-2}(\tau)$ which is also equal to a single component, namely the link pattern π'_{max} connecting $(2i, 2i + 1)$, and $(1, 2n)$; (iii)

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$t = 1$ corresponding to the sum rule $\sum_{\pi \in LP_{2n}} \Psi_\pi = K(1|\tau)$. From the definition (4.33), it is readily computed as

$$K(t|\tau) = \oint \cdots \oint \prod_{m=1}^{n} \frac{du_m(1 + tu_m)}{2\pi i u_m^{2m-1}} \left[\prod_{1 \leq \ell \leq m \leq n} (1 - u_\ell u_m) \right.$$

$$\left. \times \prod_{1 \leq \ell < m \leq n} (u_m - u_\ell)(1 + \tau u_m + u_\ell u_m)(\tau + u_\ell + u_m) \right] \tag{5.11}$$

We now make use of the following lemma, first conjectured in [9] and then proved by Zeilberger in [21].

$$\left\{ \prod_{1 \leq \ell \leq m \leq n} (1 - u_\ell u_m) \mathcal{A} \left(\prod_{m=1}^{n} u_m^{2-2m} \prod_{1 \leq \ell < m \leq n} (1 + \tau u_m + u_\ell u_m) \right) \right\}_{<0}$$

$$= \prod_{1 \leq \ell < m \leq n} (u_m^{-1} - u_\ell^{-1})(\tau + u_\ell^{-1} + u_m^{-1}) = \Delta(u^{-1}(\tau + u^{-1})) \tag{5.12}$$

where the subscript < 0 means that we retain in the corresponding multiple Laurent series of the u_m the terms with only non-positive powers, and the symbol $\mathcal{A}$ means as before antisymmetrization with respect to the permutations of variables: $\mathcal{A}(f)(u_1, \ldots, u_n) \equiv \sum_{\sigma \in S_n} (-1)^\sigma f(u_{\sigma(1)}, \ldots, u_{\sigma(n)})$. Noting that $\Delta(u) = \mathcal{A}(\prod_{m=1}^{n} u_m^{m-1})$, and that $\oint \mathcal{A}(f)g = \oint f \mathcal{A}(g)$, and applying the lemma to (5.11), we get

$$K(t|\tau) = \oint \cdots \oint \prod_{m=1}^{n} \frac{du_m(1 + tu_m)u_m^{m-1}}{2\pi i u_m} \prod_{1 \leq \ell < m \leq n} (\tau + u_\ell + u_m) \, \mathcal{A}\left(\prod_{m=1}^{n} u_m^{1-m}(\tau + u_m^{-1})^{m-1} \right)$$

$$= \oint \cdots \oint \prod_{m=1}^{n} \frac{du_m}{2\pi i u_m}(1 + tu_m)u_m^{1-m}(\tau + u_m^{-1})^{m-1} \Delta(u(\tau + u))$$

$$= \det_{1 \leq \ell, m \leq n} \left(\oint \frac{du}{2\pi i u}(1 + tu)u^{\ell-m}(\tau + u^{-1})^{m-1}(\tau + u)^{\ell-1} \right) \tag{5.13}$$

where in the last step we have used the multilinearity of the Vandermonde determinant to rewrite the whole multiple integral as a determinant of single integrals. This finally yields

$$K(t|\tau) = \det_{1 \leq \ell, m \leq n} (f_{\ell,m}(t|\tau))$$

$$f_{\ell,m}(t|\tau) = \sum_r \tau^{2\ell+2m-3-2r} \binom{\ell-1}{r-\ell} \left(\tau \binom{m-1}{r-m} + t \binom{m-1}{r+1-m} \right) \tag{5.14}$$
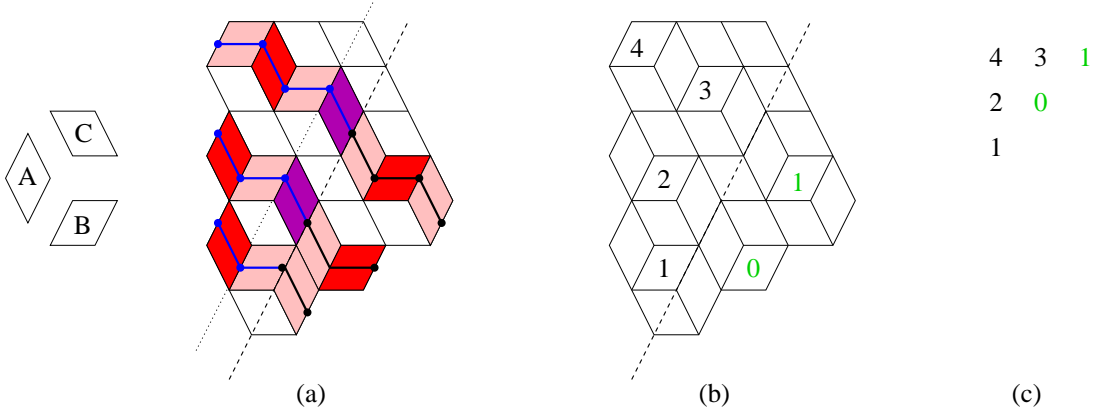
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Fig. 6: (a) The fundamental domain of a modified CSTCPP and its NILP description (blue and black paths) and (b) the correspondence (heights of the box piles) to triangular arrays of integers (c). Colors are related to weights: red means a weight of τ, purple $1/t$, green x. With these conventions, (a) receives a weight $\tau^3 \times t^{-2} \times \tau^2$ (times a global factor of $t^{n-1} = t^3$), while (b) and (c) have the weight x^2.

In order to reconnect with the results of [7], it is convenient to rewrite $K(t|\tau)$, by shifting all indices: $\ell = j + 1$, $m = i + 1$, $r = s + 1$, and noting that the first row/column do not contribute to the determinant:

$$K(t|\tau) = \det_{1 \leq i, j \leq n-1} \left[\sum_s \tau^{2i+2j-2s} \binom{i}{2i-s} \left(t\tau \binom{j}{2j-s+1} + \binom{j}{2j-s} \right) \right] \quad (5.15)$$

For generic t, the polynomials $K(t|\tau)$ correspond to a refined τ, t-enumeration of CSTCPP$^\triangle$, which can be described as follows. In the NILP formulation of the CSTCPP$^\triangle$ the CSTCPP$^\triangle$ are described by paths made by two orientations of lozenges in a fundamental domain of the CSTCPP$^\triangle$: these are the (colored) lozenges of types A, B on Fig. 6(a). In Ref. [7], these paths were viewed as pairs of paths sharing their arrival point, the paths on one side being one step longer than on the other side: the set of paths below the dashed line of Fig. 6(a) was identified as the NILP in bijection with TSSCPP (paths represented in black), while that above was viewed as an augmented one, with one more last step in each path (paths represented in blue, with the last step taking place in the strip just above the dashed line).

At each step of the paths, lozenges of type A above the diagonal dashed line and B below (in red on the figure) are given a weight of τ in $K(t|\tau)$, *except* the last step of the longer paths (in the strip just above the dashed line), where a factor t is given to lozenges of type B; it is however more convenient to consider for this last step that the factor of τ is replaced by $1/t$ (purple lozenges on the figure) up to global multiplication by t^{n-1}.

Let us now discuss various specializations of $K(t|\tau)$.

As announced above, at $t = 0$, we find the maximal component

$$\Psi_{\pi_{max}}(\tau) = K(0|\tau) = \det_{1 \le i,j \le n-1} \left[\sum_s \tau^{2i+2j-2s} \binom{i}{2i-s} \binom{j}{2j-s} \right] \qquad (5.16)$$

This matches the expression conjectured in [7] (conjecture 3, Eq. (5.1)). Likewise, at $t \to \infty$, we find:

$$\Psi_{\pi'_{max}}(\tau) = \lim_{t \to \infty} \frac{1}{t^n} K(t|\tau) = \det_{1 \le i,j \le n-1} \left[\sum_s \tau^{2i+2j-2s+1} \binom{i}{2i-s} \binom{j}{2j-s+1} \right] \qquad (5.17)$$

We also obtain at $t = 1$ the sum rule for the components of Ψ:

$$\sum_{\pi \in LP_{2n}} \Psi_\pi(\tau) = \det_{1 \le i,j \le n-1} \left[\sum_s \tau^{2i+2j-2s} \binom{i}{2i-s} \left(\tau \binom{j}{2j-s+1} + \binom{j}{2j-s} \right) \right] \qquad (5.18)$$

which matches the conjectured expression found in Ref. [7] (Eq. (4.10)).

Finally, one more identification is of some interest. Setting $t = \tau^{-1}$ we find the generating function $T_n(x = \tau^2, 1)$ of [22]. The latter has several intepretations. One of them is the following: consider triangular arrays of non-negative integers a_{ij}, $i, j \ge 1$, $i + j \le n$, with weakly decreasing rows and columns and such that $a_{i1} \le n - i + 1$ for all i. These arrays turn out to be in bijection with CSTCPP$^\triangle$, see Fig. 6(b); to produce $T_n(x, 1)$ one gives a weight x to parts a_{ij} such that $a_{ij} \le j - 1$, see Fig. 6(c). Via the bijection this corresponds in terms of plane partitions to lozenges of type B that are below the diagonal. We now show that this is the same weight that is given to CSTCPP$^\triangle$ in $K(\tau^{-1}|\tau)$. When $t = \tau^{-1}$ all red/purple lozenges get a weight of τ. Call n_{ab} the number of lozenges of type a in region b where $a = A, B, C$ and $b = \uparrow, \downarrow$ corresponds to above/below the diagonal dashed line. Then the weight for CSTCPP$^\triangle$ is $\tau^{n_{A\uparrow} + n_{B\downarrow}}$, whereas the weight for triangular arrays is $x^{n_{B\downarrow}}$. Now the number of tiles of each orientation is fixed (independent of the choice of plane partition): in particular $n_{A\uparrow} + n_{A\downarrow} = n(n+1)/2 - 1$. Furthermore the number of tiles of the first 2 types in each region is also fixed: $n_{A\uparrow} + n_{B\uparrow} = n(n-1)/2$. We conclude immediately that $n_{A\uparrow} + n_{B\downarrow} = 2n_{B\downarrow} + n - 1$, which allows us to identify the weights taking into account the prefactor t^{n-1} and the equality $x = \tau^2$.

$T_n(x, 1)$ is also conjectured to be the x-enumeration of VSASM where a weight x is given to each pair of -1s (conj. 3.2 of [22]). In the current framework there is no obvious explanation of this coincidence.

6. The odd case

6.1. General solution

The link patterns in size $2n + 1$ are obtained bijectively from those in size one more $2n + 2$, by simply erasing the rightmost arch and suppressing the rightmost endpoint $2n + 2$, so that the opening point of the rightmost arch remains isolated and unmatched. The component of Ψ in size $2n + 1$ can then be obtained from the corresponding one in size $2n + 2$ by setting $z_N = 0$. There is of course another bijection which consists on the contrary in erasing the leftmost arch and endpoint, so that the closing point of the leftmost arch remains unmatched. This time setting $z_1 = \infty$ allows to obtain Ψ in odd size $2n + 1$ from even size $2n + 2$. Since the qKZ equation is not quite left-right symmetric these two routes lead to different expressions.

Here we use the second route, having in mind that eventually we shall apply left-right symmetry as before to express everything in terms of closings instead of openings. We start with Eq. (4.31) at $N = 2n + 2$ with $a_1 = 1$ and integrate over w_1; we set $z_1 = \infty$ at this stage and obtain after shifting one step the indices of the a's, w's and z's:

$$
\Psi'_{a_1,\ldots,a_n}(z_1,\ldots,z_N) = \prod_{1 \le i < j \le N} (qz_i - q^{-1}z_j)(q^4 - z_i z_j) \oint \cdots \oint \prod_{m=1}^{n} \frac{dw_\ell}{2\pi i}
$$
$$
\frac{\prod_{1 \le \ell < m \le n}(w_m - w_\ell)(qw_\ell - q^{-1}w_m)(q^2 - w_\ell w_m) \prod_{1 \le \ell \le m \le n}(q^4 - w_\ell w_m)}{\prod_{\ell=1}^{n} \prod_{i=1}^{N}(q^4 - w_\ell z_i) \prod_{i=1}^{a_\ell}(w_\ell - z_i) \prod_{i=a_\ell+1}^{N}(qw_\ell - q^{-1}z_i)} \quad (6.1)
$$

that is, exactly the same expression as Eq. (4.31), but in which now $N = 2n+1$. It provides us with Ψ in odd size in terms of the openings a_i, with the convention that the closing of the former leftmost arch is unmatched (the opening, i.e. the leftmost point, being erased).

We now use left-right symmetry in the form of the inversion property (3.4), to obtain the component $\Psi'_{b_1,\ldots,b_n}$ defined in terms of the n closings b_i counted from right to left, with the convention that the opening of the former rightmost arch is unmatched. We simply take Eq. (6.1) and substitute a_i with b_i and z_i with q^3/z_i. Note that $b_i \le 2i - 1$ for closings to the right of the unmatched point and $b_i \le 2i$ for closings to its left. We can now take the homogeneous limit $z_i = 1$ and obtain

$$
K'_{b_1,b_2,\ldots,b_n}(\tau) = \oint \cdots \oint \prod_{m=1}^{n} \frac{du_m(1 + \tau u_m + u_m^2)}{2\pi i u_m^{b_m}} \left[\prod_{1 \le \ell \le m \le n} (1 - u_\ell u_m) \right.
$$
$$
\left. \times \prod_{1 \le \ell < m \le n} (u_m - u_\ell)(1 + \tau u_m + u_\ell u_m)(\tau + u_\ell + u_m) \right] \quad (6.2)
$$

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6.2. Solutions at $\tau \to 0$ and large τ

Repeating the calculations of Sects. 5.1 and 5.2, and using the same changes of variables, we arrive easily at the following formulas, respectively for $\tau \to 0$:

$$K'_{b_1,\ldots,b_n}(\tau) \sim \tau^{n(n-1)-\sum(b_i-1)} \det_{1\leq\ell,m\leq n} \binom{m-1}{b_\ell - m} \tag{6.3}$$

and $\tau \to \infty$:

$$K'_{b_1,\ldots,b_n}(\tau) \sim \tau^{\sum(b_i-1)} \det_{1\leq\ell,m\leq n} \binom{\ell}{b_\ell - m} \tag{6.4}$$

Note that the determinants (6.3) vanish when some of the b_j's attains $2j$, hence the correct small τ behaviour is actually subleading in those cases. The answer (6.3) is accurate only if the unmatched point of the corresponding link pattern is at position 1 (in which case all $b_i \leq 2i - 1$). In general, the correct answer depends on the position of this point, and requires a further expansion in powers of τ, that we shall omit here.

Note also that the size $2n + 1$ large τ solution (6.4) is related to the one in size $2n + 2$ (5.6) via:

$$K'_{b_1,\ldots,b_n}(\tau) \sim K_{1,1+b_1,\ldots,1+b_n}(\tau) \tag{6.5}$$

Indeed, as mentioned above, the odd size $2n + 1$ link patterns π' are obtained from those of size one more $2n + 2$ π by removing the rightmost arch and leaving its beginning point unmatched, so arch closings indeed get shifted by one unit in this bijection, namely $b_1(\pi) = 1$, and $b_i(\pi) = b_{i-1}(\pi') + 1$ for $i = 2, \ldots, n+1$. This makes the connection between Eqs. (6.4) and (5.6) explicit.

The results (6.3)–(6.4) translate immediately into the corresponding identical small and large τ estimates for $\Psi_\pi(\tau)$, as the change of basis from arch openings to link patterns is directly inherited from that of the even case $2n + 2$.

6.3. Generalized sum rule

An analogous statement as that of Sect. 5.3 may be derived for the case of odd size $2n + 1$, in which we now have to sum over "arch closing" basis elements $K'_{b_1,\ldots,b_n}$ with $b_i = 2i - \epsilon_i$, $\epsilon_i \in \{0, 1\}$. The generalized sum rule now reads:

$$K'(t|\tau) \equiv \sum_{\epsilon_i \in \{0,1\}} t^{\sum \epsilon_i} K'_{2-\epsilon_1, 4-\epsilon_2, \ldots, 2n-\epsilon_n}(\tau) \tag{6.6}$$

and reproduces $\sum_{\pi \in LP_{2n+1}} \Psi_\pi$ at $t = 1$.

Repeating exactly the same steps as those of Section 5.3, with now an extra factor of $\prod_{m=1}^{n}(1 + \tau u_m + u_m^2)/u_m$, we easily arrive at:

$$K'(t|\tau) = \det_{1\leq\ell,m\leq n}\left[\oint \frac{du}{2\pi i u^2}(1 + u^{-1}(\tau + u^{-1}))(1 + tu)u^{\ell-m}(\tau + u^{-1})^{m-1}(\tau + u)^{\ell-1}\right]$$
$$= \det_{1\leq\ell,m\leq n}(\phi_{\ell,m}(t|\tau) + \phi_{\ell,m+1}(t|\tau)) \tag{6.7}$$

where

$$\phi_{\ell,m} = \oint \frac{du}{2\pi i u^2}(1 + tu)u^{\ell-m}(\tau + u^{-1})^{m-1}(\tau + u)^{\ell-1}$$
$$= \sum_r \tau^{2\ell+2m-2r-4}\binom{\ell-1}{r-\ell}\left[\tau\binom{m-1}{r+1-m} + t\binom{m-1}{r+2-m}\right] \tag{6.8}$$

It is straightforward to show that

$$K'(t|\tau) = \det_{1\leq\ell,m\leq n}(g_{\ell,m}(t|\tau) + g_{\ell-1,m}(t|\tau))$$
$$= \det_{1\leq\ell,m\leq n}(g_{\ell,m}(t|\tau)) \tag{6.9}$$

where we have defined

$$g_{\ell,m}(t|\tau) = \sum_r \tau^{2\ell+2m-2r-1}\binom{\ell}{r-\ell}\left[\tau\binom{m-1}{2m-r} + t\binom{m-1}{2m-1-r}\right] \tag{6.10}$$

The proof of (6.9) is obtained by performing a term-by-term identification of $\phi_{\ell,m}+\phi_{\ell,m+1}$ with $g_{\ell-1,m}+g_{\ell,m}$, and then identifying the determinant of these entries with $\det(g_{\ell,m}(t|\tau))$ by column manipulations of the latter matrix.
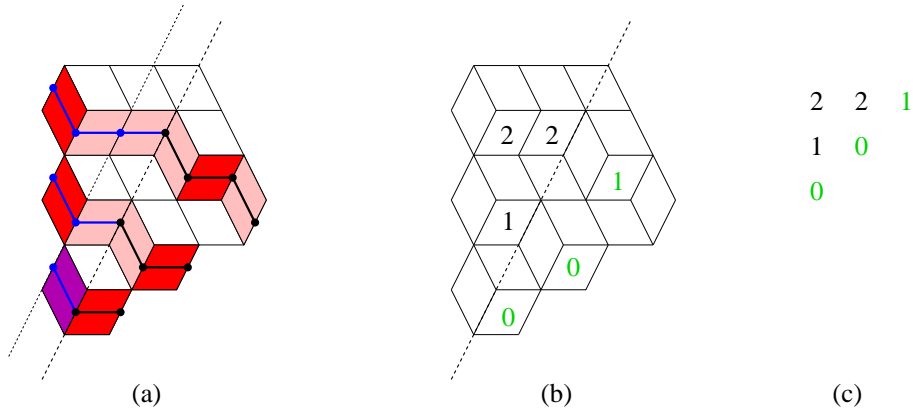

Fig. 7: (a) The fundamental domain of a CSTCPP and its NILP description (blue and black paths) and (b) the correspondence (heights of the box piles) to triangular arrays of integers (c). The color code for weights here is: red=τ, purple=t, and green=x.

The NILP formulation of CSTCPP given in Ref. [7] is similar to the even case, except this time the paths are viewed as two sets of paths of equal lengths (each in bijection with TSSCPP). The generating function $K'(t|\tau)$ corresponds to the generating polynomial of these objects, with a weight τ for the same type of tiles as in the even case, see Fig. 7, *except* in the last steps of one of the set of paths, where this weight is replaced by t.

The expression in the second line of (6.9) for $t = 1$ matches exactly the conjectured result of Ref. [7] (Eq. (4.2)), equal to the generating polynomial for weighted CSTCPP.

Likewise, at $t = 0$ one recovers the conjectured expression of Ref. [7] (conjecture 4, Eq. (5.3)), and the limit $t \to \infty$ yields the corresponding reflected link pattern (conjecture 3, Eq. (5.2)). Note that the latter is identical to the sum rule $K(\tau^{-1}|\tau)$ in even size one less $(2n)$ with $t = \tau^{-1}$, due to simple identities relating K' and K. Therefore it is also equal to the generating function $T_n(\tau^2, 1)$ of [22].

Finally at $t = \tau$ we can identify $K'(\tau|\tau)$ with the generating function $T_n(\tau^2, 0)$ defined in [22]. Once again, this is no surprise since $T_n(x, 0)$ is the generating function for triangular arrays of non-negative integers a_{ij}, $i, j \geq 1$, $i + j \leq n$, with weakly decreasing rows and columns and such that $a_{i1} \leq n - i$ for all i, which turn out to be in one-to-one correspondence with CSTCPP, see Fig. 7. As in the case of even size, one can check that the refinements are the same, i.e. that the weight x given to lozenges of one of the three types that are below the diagonal is the same weight that is given to CSTCPP in $K'(\tau|\tau)$ if one sets $x = \tau^2$. Note that once again simple identities show that $K'(\tau|\tau)$ is equal to $K(0|\tau)$, that is the component $\Psi_{\pi_{max}}(\tau)$ of size $2n$.

7. Conclusion

In this paper, we have proved various conjectures regarding the minimal polynomial solution of the qKZ equation with reflecting boundaries with q generic, expressed in the link pattern basis. This was done by writing the solutions as multiple residue integrals, modulo a triangular change of basis. As both integrals and the change of basis are completely explicit, we therefore end up with an explicit formula for each component $\Psi_\pi(z_1, ..., z_N)$. We hope these expressions will help us address the full Razumov–Stroganov conjecture which gives a conjectural interpretation for each Ψ_π in the homogeneous case $z_1 = ... = z_N = 1$ (and $q = -e^{i\pi/3}$), and hopefully come up with a more general combinatorial interpretation of the polynomials $\Psi_\pi(\tau)$ in the homogeneous case for generic q. Note that

we have now a numerical recipe for calculating the $\Psi_\pi(\tau)$, via the explicit inversion of the change of basis, and an explicit generation of the integrals.

As stressed and proved in this paper, the above change of basis is independent of the details of the boundary conditions imposed in addition to the main exchange relation $t_i\Psi = (e_i - \tau)\Psi$. These details are simply reflected by the insertion of some specific symmetric function F in the definition of the integrals. The techniques of the present paper may therefore presumably be adapted to include the other boundary conditions considered in [23], parametrized by the root systems of classical Lie algebras.

As shown in [8], generalizations of the Razumov–Stroganov sum rule have been obtained and proved in the case of the level 1 qKZ equation pertaining to higher rank (sl_k) algebras at specific values of the parameter q ($q = -e^{\frac{i\pi}{k+1}}$ and $q = -1$). We believe the construction of the present paper may be generalized to these cases, and may lead to new combinatorial interpretations, such as generalizations of Plane Partitions.

Another direction of future research is to revisit the model of *crossing* loops considered in [24,25,26], which is based on the Brauer algebra, and obtain integral formulae in the same spirit as those of the present work. It would then be particularly interesting to understand their interrelation with the geometry of the Brauer loop scheme.

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